

K-MODULI OF CUBIC HYPERSURFACES

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ABSTRACT. We prove that the K-moduli space of cubic n -folds is isomorphic to their GIT moduli space. In particular, all cubic hypersurfaces with at worst ordinary double points are K-stable. The main ingredients are a series of refined inequalities relating local invariants of singularities and the semistability of the tangent sheaf of K-semistable Fano varieties.

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1. INTRODUCTION

K-stability, introduced by differential geometers [Tia97, Don02], is an algebraic condition characterizing the existence of a Kähler–Einstein metric on a Fano variety. The algebraic theory of K-stability has undergone substantial development over the past decade. One of its chief achievements is the construction of projective moduli spaces parametrizing K-polystable Fano varieties, known as *K-moduli spaces*; see, e.g., [Xu25].

Despite this general theory, describing even a single component of a K-moduli space is rarely feasible, especially in higher dimensions. The *moduli continuity method* provides an effective approach to the explicit study of K-moduli spaces of Fano varieties with large anticanonical volume. It has been successfully applied to del Pezzo surfaces (cf. [OSS16]) and various families of Fano threefolds (cf. [ADL23, LX19, LZ25, Zha26, KLPZ26]). The method relies on a priori control of the singularities of K-semistable Fano varieties, notably through the ODP *gap conjecture* (cf. [XZ26, Problem 33]), which is known in dimensions at most three (cf. [LX19, Theorem 1.3]) and, in arbitrary dimension, for lci singularities (cf. [Liu22, Theorem 1.3]). Consequently, relatively few higher-dimensional examples are currently understood.

Higher-dimensional Fano hypersurfaces form some of the most basic and explicit families of Fano varieties with nontrivial moduli. Cubic hypersurfaces are *del Pezzo varieties*, namely Fano varieties of coindex two. By the classification of del Pezzo varieties, the cases with nontrivial moduli and sufficiently large anticanonical volume are precisely the cubic hypersurfaces and complete intersections of two quadrics. The K-moduli spaces of cubic hypersurfaces are known in dimensions at most four (cf. [OSS16, LX19, Liu22]), while the K-moduli spaces of complete intersections of two quadrics were described in all dimensions (cf. [SS17]), where the large anticanonical volume plays

a crucial role in controlling the singularities of K-semistable limits. Together, these results provide evidence for the expectation that the K-moduli spaces of cubic hypersurfaces agree with their GIT moduli spaces.

In this paper, we confirm this expectation for cubic hypersurfaces. Let $\mathcal{M}_{\text{cubic}}^K$ be the irreducible component of $\mathcal{M}_{n,V}^K$ parametrizing K-semistable Fano varieties of dimension n with anticanonical volume $V := 3 \cdot (n-1)^n$ that admit a $\mathbb{Q}$ -Gorenstein deformation to a smooth cubic hypersurface in $\mathbb{P}^{n+1}$, and let $\mathcal{M}_{\text{cubic}}^{\text{GIT}}$ denote the GIT quotient stack of cubic hypersurfaces.

Theorem 1.1. *Every K-semistable Fano variety parametrized by $\mathcal{M}_{\text{cubic}}^K$ is isomorphic to a cubic hypersurface in $\mathbb{P}^{n+1}$. In particular:*

- (1) *a cubic hypersurface $X \subseteq \mathbb{P}^{n+1}$ is K-(semi/poly)stable if and only if it is GIT-(semi/poly)stable;*
- (2) *there is a natural isomorphism*

$$\mathcal{M}_{\text{cubic}}^K \xrightarrow{\sim} \mathcal{M}_{\text{cubic}}^{\text{GIT}},$$

which induces an isomorphism between their good moduli spaces $\overline{\mathfrak{M}}_{\text{cubic}}^K \simeq \overline{\mathfrak{M}}_{\text{cubic}}^{\text{GIT}}$;

- (3) *$\mathcal{M}_{\text{cubic}}^K$ is a smooth connected component of $\mathcal{M}_{n,V}^K$.*

K-moduli theory not only provides a framework for constructing moduli spaces but also offers powerful tools for verifying the K-stability of explicit Fano varieties. Combining Theorem 1.1 with the GIT stability of cubic hypersurfaces yields the following consequence.

Corollary 1.2. *Every cubic hypersurface with at worst ordinary double points is K-stable. In particular, every smooth cubic hypersurface admits a Kähler–Einstein metric.*

1.1. Sketch of proofs. The key to proving Theorem 1.1 is to show that a K-semistable limit X of cubic n -folds remains a cubic hypersurface in $\mathbb{P}^{n+1}$. By the classification of del Pezzo varieties, this reduces to proving that the $\mathbb{Q}$ -Cartier Weil divisor L satisfying $-K_X \sim_{\mathbb{Q}} (n-1)L$ is in fact Cartier. The main idea is straightforward: we study the global geometry of the linear series $|L|$, using inequalities derived from normalized volume bounds as local input. In particular, $|L|$ defines a rational map $\phi_L: X \dashrightarrow Y \subseteq \mathbb{P}^{n+1}$. Let $p: W \rightarrow X$ denote its normalized graph.

The argument follows a trichotomy according to the dimension of the image Y . If $\dim Y = 1$, we construct a family of divisors in $|L|$ containing components of high multiplicity that sweep out X . These divisors give an upper bound on $\alpha(X)$ that contradicts the lower bound imposed by K-semistability.

When $2 \leq \dim Y \leq n-1$, the key is to show that Y is a variety of minimal degree. An essential input is the slope semistability of the tangent sheaf on a resolution, with respect to the pullback of the anticanonical polarization, afforded by K-semistability of X . The resulting slope inequality, together with vanishing theorems and a Koszul resolution, shows that a general complete intersection curve section is rational. A second Koszul resolution computes the number of sections of the restricted polarization, showing that this curve is a rational normal curve. Thus Y has minimal degree. The classification in [EH87] then allows us to construct divisors that violate the valuative inequalities imposed by K-semistability.

The remaining case $\dim Y = n$ is governed by local constraints on singularities. The degree bound reduces the possibilities for Y to a quadric or a cubic hypersurface. In the quadric case, the geometry of Y gives strong lower bounds on the S -invariants, which K-semistability converts

into lower bounds on local log canonical thresholds on the index-one cover. These bounds force the cover to be a local complete intersection, contradicting the normalized-volume bound. In the cubic case, equality of the top intersection numbers forces the fixed part to vanish, and L is therefore Cartier.

Comparison with lower-dimensional cases. It is worth comparing this argument with those for the lower-dimensional cases. In [LX19], the authors prove the ODP *gap conjecture* in dimension three. In higher dimensions, this conjecture is only known for lci singularities (cf. [Liu22, Theorem 1.3]). Thus, the Cartierness of L follows immediately in dimension three. For cubic fourfolds, [Liu22] shows that a K-semistable limit X satisfies $-K_X \sim_{\mathbb{Q}} 3L$ for a $\mathbb{Q}$ -Cartier Weil divisor L of Cartier index at most two. A general member of $|2L|$ avoids the non-Cartier locus of L , and adjunction gives a degree-6 polarized K3 surface $(S, L|_S)$ by intersecting a general member of $|2L|$ with one of $|L|$. The K-semistability of X then reduces the possible K3 surfaces to the case of a $(2, 3)$ -complete intersection in $\mathbb{P}^4$, which can be handled by the lci version of the ODP gap conjecture.

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The authors do not claim any traditional academic credit for this paper.

This paper is not intended for publication in a traditional mathematics journal.

Story behind the paper. This paper grew out of a project involving Shitan Xu, Ni Tang, and the authors BK, YS, and MZ, which explored whether GPT-5.6 Sol could prove the main theorem in dimension five. After obtaining an initial proof generated entirely by AI, they invited JZ to join the project and help verify it. The authors subsequently checked and simplified the proof.

The authors later found an alternative proof of part of the results in the five-dimensional case and recognized that this approach might extend to higher dimensions. In attempting to extend this approach to higher dimensions, the authors encountered a gap in the argument that the image Y of the limiting linear system must be a variety of minimal degree when $2 \leq \dim Y \leq n - 1$. They presented the incomplete argument to GPT, which filled the gap and completed the proof of this assertion. An essential input introduced by GPT was the use of “slope semistability of the tangent sheaf arising from K-semistability”. The authors subsequently verified the resulting argument. This made it possible to rule out the intermediate-dimensional case, which proved to be the most difficult part of the argument in arbitrary dimension.

The authors verified the arguments and wrote the paper from scratch to highlight its central ideas. ChatGPT was also used for language editing. The authors take full responsibility for all claims and arguments presented in the paper.

An independent development. After the author obtained the proof in dimension five, JZ posted it on his personal webpage. Long Pan subsequently contacted JZ, suggesting that the argument could be extended to dimension six and informing JZ that he had completed such a proof.

After the author obtained a proof in arbitrary dimension, JZ informed Long Pan of their result. Long Pan told the authors that, in the meantime, he, Zhiyuan Li, and Haoyu Wu had also obtained a proof in arbitrary dimension using GPT. A brief exchange indicated that the proof found by Li–Pan–Wu likewise used “slope semistability arising from K-semistability as an essential new input”.

This experience illustrates how different groups presenting the same mathematical problem to GPT can arrive at similar arguments with the same essential ingredients.

2. PRELIMINARIES

In this section, we recall basic definitions and results that will be used in what follows. We work over $\mathbb{C}$, and all varieties that appear in this paper are assumed to be normal and projective.

2.1. K-stability. A variety X is called $\mathbb{Q}$ -Fano if the anti-canonical divisor $-K_X$ is an ample $\mathbb{Q}$ -Cartier divisor and X has klt singularities. The K-(semi/poly)stability of a $\mathbb{Q}$ -Fano variety X is determined by the sign and vanishing behavior of the *Futaki invariant* associated to all *test configurations* of X , as originally introduced in [Tia97, Don02]. For the purposes of this paper, it is more convenient to consider valuative formulations of these definitions.

Definition 2.1 (Valuative invariants [Fuj19, Li17, BJ20]). Let X be an n -dimensional $\mathbb{Q}$ -Fano variety. We say that E is a prime divisor over X if there exists a projective birational morphism $\pi: Y \rightarrow X$ such that E is a prime divisor on Y . We define

- the *log discrepancy* of E

$$A_X(E) := 1 + \text{coeff}_E(K_Y - \pi^*K_X);$$

- the *pseudo-effective threshold* of E

$$T_X(E) := \sup_{D \geq 0, D \sim -K_X} \text{ord}_E(D);$$

- the *S-invariant* of E

$$S_X(E) := \frac{1}{(-K_X)^n} \int_0^\infty \text{vol}_Y(\pi^*(-K_X) - tE) dt;$$

- the *δ -invariant* of X

$$\delta(X) := \inf_E \frac{A_X(E)}{S_X(E)},$$

where the infimum runs over all prime divisors E over X ;

- and the *α -invariant* of X

$$\alpha(X) := \inf \{ \text{lct}(X, D) \mid 0 \leq D \sim_{\mathbb{Q}} -K_X \}.$$

We will frequently use the following characterization of K-(semi/poly)stability and a numerical bound for K-semistability.

Theorem 2.2 (Valuative criterion for K-(semi/poly)stability, [Fuj19, Li17, BX19, LWX21]). A $\mathbb{Q}$ -Fano variety X is

- (1) *K-semistable if and only if $\delta(X) \geq 1$;*
- (2) *K-stable if and only if $\delta(X) > 1$;*
- (3) *K-polystable if and only if X is K-semistable and any $\mathbb{G}_m$ -equivariant K-semistable degeneration of X is isomorphic to X .*

Theorem 2.3 ([FO18]). Let X be an n -dimensional $\mathbb{Q}$ -Fano variety. If X is K-semistable, then $\alpha(X) \geq \frac{1}{n+1}$.

2.2. K-moduli. The following result, usually called the K-moduli theorem, is due to many authors [ABHLX20, BHLLX21, BLX22, BX19, CP21, Jia20, LWX21, LXZ22, Xu20, XZ20, XZ21].

Theorem 2.4 (K-moduli theorem). *Fix two numerical invariants $n \in \mathbb{N}$ and $V \in \mathbb{Q}_{>0}$. The moduli functor $\mathcal{M}_{n,V}^K$ sending a base scheme S to*

$$\left\{ \mathcal{X}/S \left| \begin{array}{l} \mathcal{X} \rightarrow S \text{ is a proper flat morphism, each geometric fiber} \\ \mathcal{X}_{\bar{s}} \text{ is an } n\text{-dimensional } K\text{-semistable } \mathbb{Q}\text{-Fano variety of} \\ \text{volume } V, \text{ and } \mathcal{X} \rightarrow S \text{ satisfies Kollár's condition} \end{array} \right. \right\}$$

is represented by an Artin stack, still denoted by $\mathcal{M}_{n,V}^K$, of finite type over $\mathbb{C}$ with affine diagonal. The $\mathbb{C}$ -points of $\mathcal{M}_{n,V}^K$ parameterize K -semistable $\mathbb{Q}$ -Fano varieties X of dimension n and volume V . Moreover, the Artin stack $\mathcal{M}_{n,V}^K$ admits a good moduli space $\overline{\mathcal{M}}_{n,V}^K$, which is a projective scheme whose $\mathbb{C}$ -points parameterize K -polystable $\mathbb{Q}$ -Fano varieties. The CM $\mathbb{Q}$ -line bundle λ_{CM} on $\mathcal{M}_{n,V}^K$ descends to an ample $\mathbb{Q}$ -line bundle Λ_{CM} on $\overline{\mathcal{M}}_{n,V}^K$.

2.3. Singularities and normalized volumes. Let $x \in X$ be a klt singularity. Introduced in [Li18], the *normalized volume* is a function defined on the space of valuations centered at x ; for a valuation centered at x , the normalized volume is defined in terms of two quantities: the volume and the log discrepancy. For the purposes of this paper, it suffices to consider divisorial valuations given by prime divisors centered at x .

Definition 2.5 ([Li18, ELS03]). Let $x \in X$ be an n -dimensional klt singularity. Let E be a prime divisor over X centered at x . The *volume* of E over $(x \in X)$ is defined to be

$$\text{vol}_{X,x}(E) := \limsup_{m \rightarrow \infty} \frac{\ell(\mathcal{O}_{X,x}/\mathfrak{a}_m(E))}{m^n/n!},$$

where

$$\mathfrak{a}_m(E) := \{f \in \mathcal{O}_{X,x} \mid \text{ord}_E(f) \geq m\},$$

and ℓ denotes the length of an Artinian module. The *normalized volume* of E over $(x \in X)$ is defined as

$$\widehat{\text{vol}}_{X,x}(E) := A_X(E)^n \cdot \text{vol}_{X,x}(E).$$

The *local volume* of $x \in X$ is defined as

$$\widehat{\text{vol}}(x, X) := \inf_E \widehat{\text{vol}}_{X,x}(E),$$

with the infimum taken over all prime divisors E over X centered at x .

Theorem 2.6 (cf. [Liu18, LLX18]). *Let $x \in X$ be an n -dimensional klt singularity. Then*

$$\widehat{\text{vol}}(X, x) = \inf_{\mathfrak{a}: \mathfrak{m}_x\text{-primary}} \text{lct}(X; \mathfrak{a})^n \text{mult}(\mathfrak{a}).$$

2.4. K-semistable limits of cubic hypersurfaces. We recall the properties of K-semistable limits that will be used below.

Let X be a K-semistable $\mathbb{Q}$ -Fano variety admitting a $\mathbb{Q}$ -Gorenstein smoothing $\pi: \mathcal{X} \rightarrow T$ over a smooth pointed curve $0 \in T$, with $\mathcal{X}_0 \simeq X$ and every fiber $\mathcal{X}_t$ over $t \in T^\circ := T \setminus \{0\}$ a (K-semistable) smooth cubic n -fold. After a finite base change, we may assume that $\mathcal{X}^\circ := \mathcal{X}|_{T^\circ}$ is a cubic hypersurface in $\mathbb{P}_{T^\circ}^{n+1}$. Let $\mathcal{L}^\circ$ be the restriction of $\mathcal{O}_{\mathbb{P}_{T^\circ}^{n+1}}(1)$ to $\mathcal{X}^\circ$, and let $\mathcal{L}$ be the closure of $\mathcal{L}^\circ$ in $\mathcal{X}$ as a Weil divisor. Since $-K_{\mathcal{X}^\circ} \sim (n-1)\mathcal{L}^\circ$ on $\mathcal{X}^\circ$, we have $-K_{\mathcal{X}} \sim_{\mathbb{Q}} (n-1)\mathcal{L}$; in

particular, $\mathcal{L}$ is $\mathbb{Q}$ -Cartier. Under the identification $\mathcal{X}_0 = X$, set $L := \mathcal{L}|_{\mathcal{X}_0}$. Then L is a $\mathbb{Q}$ -Cartier Weil divisor on X ; in particular, $\mathcal{O}_X(L)$ is a divisorial sheaf.

Theorem 2.7 (cf. [Liu22, Theorem 3.1]). *The following statements hold:*

- (1) $\mathcal{O}_X(L)$ is Cohen–Macaulay;
- (2) $2L$ is Cartier; in particular, X is Gorenstein canonical; and
- (3) $H^i(X, \mathcal{O}_X(mL)) = 0$ for all $i > 0$ and $m \geq 0$, and

$$h^0(X, \mathcal{O}_X(mL)) = h^0(\mathcal{X}_t, \mathcal{O}_{\mathcal{X}_t}(m))$$

for all $t \in T^\circ$.

We will also use the following lemmas.

Lemma 2.8. *Keep the notation above. Suppose that L has Cartier index two at $x \in X$. Let*

$$\phi: (\tilde{x} \in \tilde{X}) \longrightarrow (x \in X)$$

be the index-one cover with respect to L . Then

- (1) $\widehat{\text{vol}}(\tilde{x}, \tilde{X}) \geq \frac{6 \cdot (n-1)^n \cdot n^n}{(n+1)^n}$;
- (2) $(\tilde{x} \in \tilde{X})$ is not lci.

Proof. By [XZ21, Theorem 1.3] and [Liu18, Theorem 2], we have

$$\widehat{\text{vol}}(\tilde{x}, \tilde{X}) = 2\widehat{\text{vol}}(x, X) \geq 2 \left(\frac{n}{n+1} \right)^n \text{vol}(X) = \frac{6 \cdot (n-1)^n \cdot n^n}{(n+1)^n},$$

proving (1).

Suppose that $\tilde{X}$ is smooth at $\tilde{x}$. The cover is a degree-two quasi-étale cover, and X is its quotient by μ_2 . Let r be the dimension of the nontrivial eigenspace of the μ_2 -action on $T_{\tilde{x}}\tilde{X}$. Then an analytic neighborhood of x is of the form $\mathbb{A}^r/\mu_2 \times \mathbb{A}^{n-r}$. Since the cover is quasi-étale, $r \geq 2$. Suppose that $r \geq 3$. After taking $n-r$ transverse general hyperplane sections, the resulting singularity remains smoothable, contradicting [Sch71, Theorem 3]. Thus $r = 2$, which implies that x is a hypersurface singularity. By [Rob76, Theorem (ii)], L must be Cartier at x , contradicting the assumption that L is not Cartier at x . Thus $\tilde{X}$ is not smooth at $\tilde{x}$.

If $(\tilde{x} \in \tilde{X})$ were lci, then [Liu22, Theorem 1.3] would give $\widehat{\text{vol}}(\tilde{x}, \tilde{X}) \leq 2 \cdot (n-1)^n < \frac{6 \cdot (n-1)^n \cdot n^n}{(n+1)^n}$, contradicting (1). This proves (2). $\square$

For the following lemma, let X be a normal $\mathbb{Q}$ -Gorenstein projective variety over $\mathbb{C}$, and let L be an integral Weil divisor such that $2L$ is Cartier and $V := H^0(X, \mathcal{O}_X(L)) \neq 0$. Let $p: W \rightarrow X$ be the normalized graph of the rational map defined by V , and write $p^*L = L_W + E$, where L_W is the basepoint-free Cartier moving part and $E \geq 0$.

Fix a point $x \in X$ at which L has Cartier index two. Let $\mathfrak{J} \subset \mathcal{O}_{\tilde{X}}$ be the base ideal of the pullback of V to the index-one cover. The following lemma compares the log discrepancies and the vanishing orders of E on X and its index-one cover.

Lemma 2.9. *Let w be a divisorial valuation of $K(\tilde{X})$ whose restriction satisfies $w|_{K(X)} = c \cdot v$ for a normalized divisorial valuation v of $K(X)$ and $c \in \{1, 2\}$. Then*

$$A_{\tilde{X}}(w) = cA_X(v), \quad w(\mathfrak{J}) = cv(E).$$

Proof. Write $\mathfrak{J} := \text{Im}(\text{Sym}^2 V \otimes \mathcal{O}_X(-2L) \rightarrow \mathcal{O}_X)$. Then $p^{-1}\mathfrak{J} \cdot \mathcal{O}_W = \mathcal{O}_W(-2E)$, and applying a divisorial valuation v yields $v(\mathfrak{J}) = 2 \text{ord}_v(E)$. Fix a basis $s_0, \dots, s_{n+1}$ of $H^0(X, \mathcal{O}_X(L))$. Take a local trivialization U of $\mathcal{O}_X(2L)$, and write $2L|_U = \text{div}(h)$ and $h_{ij} = h \cdot s_i s_j \in \mathcal{O}_U$. If z denotes the local generator on the index-one cover satisfying $z^2 = \phi^* h$, then $f_i := z \cdot \phi^* s_i$ is regular and $f_i f_j = \phi^* h_{ij}$. Hence $\mathfrak{J}^2 = (f_i f_j)_{0 \leq i, j \leq n+1} = \phi^{-1}\mathfrak{J} \cdot \mathcal{O}_{\tilde{X}}$. It follows that

$$2w(\mathfrak{J}) = w(\mathfrak{J}^2) = w(\phi^{-1}\mathfrak{J} \cdot \mathcal{O}_{\tilde{X}}) = w|_{K(X)}(\mathfrak{J}) = c \cdot v(\mathfrak{J}) = 2c \cdot v(E),$$

and hence $w(\mathfrak{J}) = c \cdot v(E)$. For the discrepancy, choose a model of $\tilde{X}$ carrying w over a compatible model of X carrying v . Since ϕ is quasi-étale, Riemann–Hurwitz along w gives

$$A_{\tilde{X}}(w) = 1 + c(A_X(v) - 1) + (c - 1) = cA_X(v). \quad \square$$

3. IMAGE OF THE LIMITING LINEAR SYSTEM

Let $n \geq 5$ be an integer, and X be a K-semistable degeneration of smooth cubic hypersurfaces in $\mathbb{P}^{n+1}$, and L be the degeneration of the restriction of $\mathcal{O}_{\mathbb{P}^{n+1}}(1)$ as a Weil divisor. The goal of this section is to prove the following result.

Proposition 3.1. *The image of the rational map defined by $|L|$ has dimension n .*

By Theorem 2.7, one has

$$-K_X \sim_{\mathbb{Q}} (n-1)L, \quad L^n = 3, \quad h^0(X, \mathcal{O}_X(L)) = n+2.$$

Let $R = \oplus_{m \geq 0} R_m$ be the section ring of L , and let

$$\phi_L: X \dashrightarrow Y := \text{Proj } \mathbb{C}[R_1] \subseteq \mathbb{P}(R_1) = \mathbb{P}^{n+1}$$

be the rational map defined by L . Then Y is integral and nondegenerate of dimension $d \leq n$. Let W be the normalization of the graph of ϕ_L , and $\nu: Y^\nu \rightarrow Y$ be the Stein factorization of $W \rightarrow Y$. Then one has the following diagram:

$$\begin{array}{ccccc} & & W & \xrightarrow{q} & Y^\nu \\ & \swarrow p & & & \searrow \nu \\ \phi_L: & X & \dashrightarrow & Y & \hookrightarrow \mathbb{P}^{n+1}. \end{array}$$

Let L_Y be the hyperplane class $\mathcal{O}_Y(1)$, and

$$L_{Y^\nu} := \nu^* L_Y, \quad L_W := q^* L_Y, \quad E := p^* L - L_W.$$

Then L_W is nef, L_{Y^ν} and $L_W + p^* L$ are ample, and E is the fixed part of the linear series $|p^* L|$. In particular, E is an effective $\mathbb{Q}$ -Cartier Weil divisor of Cartier index at most 2. Define the mixed intersection numbers by

$$a_i = ((p^* L)^{n-i} \cdot L_W^i) \quad (0 \leq i \leq n)$$

They satisfy the relation

$$a_i - a_{i+1} = ((p^* L)^{n-i-1} \cdot L_W^i \cdot E) \geq 0, \quad a_0 = 3. \quad (3.1)$$

Finally, we set

$$\kappa := \frac{n-1}{3(n+1)} \sum_{i=0}^d a_i, \quad d := \dim Y.$$

Proposition 3.2. *For every divisorial valuation v over X with $v(E) > 0$, one has*

$$S_X(v) \geq \kappa \cdot v(E).$$

Proof. Choose a log resolution $\rho: W' \rightarrow W$ such that $v = \text{ord}_G$ for a prime divisor G on W' . Set $p' := p \circ \rho$, $L_{W'} := \rho^* L_W$, and $E' := \rho^* E$, so that $p'^* L = L_{W'} + E'$. For every $0 \leq u \leq 1$, we have

$$\begin{aligned} p'^* L - u \cdot v(E) \cdot G &= p'^* L - u E' + u(E' - v(E) \cdot G) \\ &= (1-u)p'^* L + u L_{W'} + u(E' - v(E) \cdot G) \\ &\geq (1-u)p'^* L + u L_{W'}, \end{aligned}$$

where $(1-u)p'^* L + u L_{W'}$ is big and nef. Using $-K_X \sim_{\mathbb{Q}} (n-1)L$ and $\text{vol}(-K_X) = 3(n-1)^n$, we obtain

$$\begin{aligned} S_X(v) &= \frac{1}{\text{vol}(-K_X)} \int_0^\infty \text{vol}(p'^*(-K_X) - tG) dt \\ &\geq \frac{1}{3(n-1)^n} \int_0^{(n-1)v(E)} \text{vol}((n-1)p'^* L - tG) dt \\ &= \frac{(n-1)v(E)}{3(n-1)^n} \int_0^1 \text{vol}((n-1)(p'^* L - u \cdot v(E) \cdot G)) du \\ &\geq \frac{(n-1)v(E)}{3} \int_0^1 ((1-u)p'^* L + u L_{W'})^n du \\ &= \frac{(n-1)v(E)}{3} \sum_{i=0}^d a_i \binom{n}{i} \int_0^1 u^i (1-u)^{n-i} du \\ &= \frac{(n-1)v(E)}{3(n+1)} \sum_{i=0}^d a_i = \kappa \cdot v(E), \end{aligned}$$

where the penultimate equality follows from the beta integral. $\square$

Let $\pi_Y: \tilde{Y} \rightarrow Y^\nu$ be a log resolution, $\tilde{q}: W \dashrightarrow \tilde{Y}$ be the induced rational map, and $\tilde{W}$ be a log resolution of the normalization of the graph of $\tilde{q}$. Then we have the following commutative diagram:

$$\begin{array}{ccccc} & \tilde{W} & \xrightarrow{\tilde{q}} & \tilde{Y} & \\ & \downarrow \pi_W & \nearrow \tilde{q} & \downarrow \pi_Y & \\ & W & \xrightarrow{q} & Y^\nu & \\ \tilde{p} \swarrow & & & & \searrow \nu \\ X & \xrightarrow{\quad p \quad} & & Y & \hookrightarrow \mathbb{P}^{n+1}. \end{array}$$

To ease the notations, we set

$$L_{\tilde{Y}} := \pi_Y^* L_{Y^\nu}, \quad H := \tilde{p}^* L, \quad L_{\tilde{W}} := \tilde{q}^* L_{Y^\nu}, \quad \tilde{E} := \pi_W^* E.$$

Then $H = L_{\tilde{W}} + \tilde{E}$ is a big and nef $\mathbb{Q}$ -line bundle.

Lemma 3.3. *Keep the notation introduced above. Then the following statements hold:*

- (1) *the morphism $\tilde{q}$ has connected fibers;*
- (2) $h^0(\tilde{Y}, L_{\tilde{Y}}) = n + 2$.

Proof. Consider the Stein factorization $\tilde{W} \xrightarrow{g} \tilde{Y}' \xrightarrow{h} \tilde{Y}$ of $\tilde{q}$. Since π_Y is birational and q has connected fibers, the morphism $\tilde{q}$ has connected fibers over a dense open subset of $\tilde{Y}$. It follows

that the finite morphism h is birational. Since $\tilde{Y}$ is normal, Zariski's Main Theorem implies that h is an isomorphism. This proves (1). For (2), the projection formula gives

$$H^0(\tilde{Y}, L_{\tilde{Y}}) \simeq H^0(Y^\nu, L_{Y^\nu}) \simeq H^0(W, L_W) \simeq H^0(X, L).$$

Therefore, $h^0(\tilde{Y}, L_{\tilde{Y}}) = n + 2$. $\square$

Proposition 3.4. *If $2 \leq d < n$, then $Y \subseteq \mathbb{P}^{n+1}$ is a non-degenerate variety of minimal degree.*

Proof. Let $H_1, \dots, H_{d-1} \in |L_{\tilde{Y}}|$ be general divisors, and $C := \cap_i H_i$ be the smooth complete intersection curve. As

$$(L_{\tilde{Y}} \cdot C) = (L_{\tilde{Y}}^d) = \deg \nu \cdot (L_Y^d) = \deg \nu \deg Y \geq \deg Y,$$

it suffices to show that $\deg L_{\tilde{Y}}|_C = n + 2 - d$. This follows from the following lemma.

Lemma 3.5. *The C is a rational curve, and $h^0(C, L_{\tilde{Y}}|_C) = n + 3 - d$.*

Proof. As X is K-semistable and $\tilde{X} \rightarrow X$ is a log resolution, by [DGP24, p. 100, the paragraph following Equation (18)], $\Omega_{\tilde{W}}$ is slope semistable with respect to the big and nef line bundle $\tilde{p}^*(-K_X)$, and hence with respect to $H = \tilde{p}^*L$. Since $\tilde{q}$ is surjective, one has an injection $\tilde{q}^*\Omega_{\tilde{Y}} \hookrightarrow \Omega_{\tilde{W}}$, and hence the slope semistability implies that

$$\frac{(\tilde{q}^*K_{\tilde{Y}} \cdot H^{n-1})}{d} \leq \frac{(K_{\tilde{W}} \cdot H^{n-1})}{n} = \frac{-3(n-1)}{n}.$$

Since H is nef and $(H^n) = 3$, one has

$$\begin{aligned} (\tilde{q}^*(K_{\tilde{Y}} + (d-1)L_{\tilde{Y}}) \cdot H^{n-1}) &\leq -\frac{3d(n-1)}{n} + (d-1)(L_{\tilde{W}} \cdot H^{n-1}) \\ &\leq -\frac{3d(n-1)}{n} + (d-1)(H^n) \leq -\frac{3(n-d)}{n} < 0, \end{aligned}$$

where the last inequality does use the assumption $d < n$. It follows that $H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(K_{\tilde{Y}} + (d-1)L_{\tilde{Y}})) = 0$. Twisting the Koszul resolution of $\mathcal{O}_C$ by $\omega_{\tilde{Y}}((d-1)L_{\tilde{Y}})$ and using adjunction, we obtain the following locally free resolution of ω_C :

$$\begin{aligned} 0 \longrightarrow \omega_{\tilde{Y}} \longrightarrow \omega_{\tilde{Y}}(L_{\tilde{Y}})^{\oplus(d-1)} \longrightarrow \dots \longrightarrow \omega_{\tilde{Y}}((d-3)L_{\tilde{Y}})^{\oplus\binom{d-1}{2}} \\ \longrightarrow \omega_{\tilde{Y}}((d-2)L_{\tilde{Y}})^{\oplus(d-1)} \longrightarrow \omega_{\tilde{Y}}((d-1)L_{\tilde{Y}}) \longrightarrow \omega_C \longrightarrow 0. \end{aligned}$$

By the Kawamata–Viehweg vanishing theorem, $H^i(\tilde{Y}, \omega_{\tilde{Y}}(tL_{\tilde{Y}})) = 0$ for any $i > 0$ and $t > 0$. Since X is a Fano variety, it is rationally connected, and hence so are $\tilde{W}$ and $\tilde{Y}$. In particular, Serre duality and Hodge symmetry yield

$$h^{d-1}(\tilde{Y}, \omega_{\tilde{Y}}) = h^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) = h^0(\tilde{Y}, \Omega_{\tilde{Y}}^1) = 0;$$

see, e.g. [Deb01, Corollary 4.18(1)]. Then [Laz04, Example B.1.3] implies that $H^0(C, \omega_C) = 0$ and hence $C \simeq \mathbb{P}^1$. Similarly, twisting the Koszul resolution of $\mathcal{O}_C$ by $\mathcal{O}_{\tilde{Y}}(L_{\tilde{Y}})$ gives a locally free resolution of $\mathcal{O}_C(L_{\tilde{Y}}|_C)$:

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\tilde{Y}}(-(d-2)L_{\tilde{Y}}) \longrightarrow \mathcal{O}_{\tilde{Y}}(-(d-3)L_{\tilde{Y}})^{\oplus(d-1)} \longrightarrow \dots \longrightarrow \mathcal{O}_{\tilde{Y}}(-L_{\tilde{Y}})^{\oplus\binom{d-1}{2}} \\ \longrightarrow \mathcal{O}_{\tilde{Y}}^{\oplus(d-1)} \longrightarrow \mathcal{O}_{\tilde{Y}}(L_{\tilde{Y}}) \longrightarrow \mathcal{O}_C(L_{\tilde{Y}}|_C) \longrightarrow 0. \end{aligned}$$

By the Kawamata–Viehweg vanishing theorem, $H^i(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(-tL_{\tilde{Y}})) = 0$ for any $i < d$ and $t < 0$. A similar chasing-through-complexes argument (cf. [Laz04, Proposition B.1.2.]) implies that

$$h^0(C, \mathcal{O}_C(L_C)) = h^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(L_{\tilde{Y}})) - h^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}^{\oplus(d-1)}) = n + 3 - d. \quad \square$$

□

Proof of Proposition 3.1. Suppose the contrary that $d \leq n - 1$. As Y is non-degenerate, one has $d \geq 1$. If $d = 1$, then one takes $y \in Y$ be a general point. Since $\dim \mathcal{O}_{Y,y}/\mathfrak{m}_y^{n+1} = n + 1$, there is a hyperplane in $\mathbb{P}^{n+1}$ meeting Y with multiplicity $n + 1$ at y . Its pullback is a member of $|L|$ of order at least $n + 1$ along a prime divisor B on X ; as $-K_X \sim_{\mathbb{Q}} (n - 1)L$, this yields $\alpha(X) \leq \frac{1}{(n+1)(n-1)}$, contradicting the K-semistability of X .

Now we may assume that $2 \leq d \leq n$. Then $Y \subseteq \mathbb{P}^{n+1}$ is a non-degenerate variety of minimal degree by Proposition 3.4. It follows from [EH87, Theorem 1] that Y is either a (possibly singular) rational normal scroll or a cone over the Veronese surface.

Suppose that Y is a rational normal scroll $S(e_1, \dots, e_d)$, where $e_1 \geq e_2 \geq \dots \geq e_d$ and $e_1 \geq 2$. Then Y admits a family of hyperplane sections $\{H_t\}_{t \in \mathbb{P}^1}$ such that H_t contains a ruling D_t with multiplicity at least 2, and the rulings $\{D_t\}_{t \in \mathbb{P}^1}$ cover Y . Since W has only finitely many p -exceptional divisors, we may choose t such that the strict transform $D \subseteq W$ of D_t is not p -exceptional. Consequently, there exists a divisor in $|L|$ containing D as a component with multiplicity at least 2. It follows that $\alpha(X) \leq \frac{1}{2(n-1)} < \frac{1}{n+1}$, contradicting the K-semistability of X .

Next, suppose that Y is a cone over a smooth quadric surface $v_2(\mathbb{P}^2)$, where we allow the vertex to be empty. Choose a line $\ell \subseteq \mathbb{P}^2$. The nonreduced conic 2ℓ determines a hyperplane section H_ℓ of $v_2(\mathbb{P}^2)$ satisfying $H_\ell = 2D_\ell$, where $D_\ell := v_2(\ell)$. Let $\overline{H}_\ell$ and $\overline{D}_\ell$ denote their respective cones in Y . Then $\overline{H}_\ell = 2\overline{D}_\ell$. Moreover, as ℓ varies, the divisors $\overline{D}_\ell$ cover Y . Therefore, the preceding argument yields $\alpha(X) \leq \frac{1}{2(n-1)} < \frac{1}{n+1}$, contradicting the K-semistability of X .

Finally, suppose that $Y = S(0^{d-e}, 1^e)$, where $e = n + 2 - d \geq 3$, and set $k := d - e$. Then Y admits a natural log resolution

$$\rho : \hat{Y} := \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^{e-1}}(\mathcal{O}^{\oplus k} \oplus \mathcal{O}(1, 1)) \longrightarrow Y.$$

Let Z be a smooth projective variety dominating both W and $\hat{Y}$:

$$\begin{array}{ccccc} & Z & \xrightarrow{q_Z} & \hat{Y} & \xrightarrow{\rho} & Y \\ & \swarrow p_Z & & \swarrow \pi_1 & \searrow \pi_2 & \\ W & & & \mathbb{P}^1 & & \mathbb{P}^{e-1} \end{array}$$

With the notation of the above diagram, set

$$L_1 := (\pi_1 \circ q_Z)^* \mathcal{O}_{\mathbb{P}^1}(1), \quad L_2 := (\pi_2 \circ q_Z)^* \mathcal{O}_{\mathbb{P}^{e-1}}(1), \quad L_Z := (\rho \circ q_Z)^* L_Y,$$

$$H_Z := (p \circ p_Z)^* L, \quad \text{and} \quad E_Z := p_Z^* E_Z.$$

Then one has the $\mathbb{Q}$ -linear relations

$$L_Z \sim_{\mathbb{Q}} L_1 + L_2 + q_Z^* \text{Exc}(\rho), \quad H_Z \sim_{\mathbb{Q}} L_Z + E_Z \sim_{\mathbb{Q}} L_1 + L_2 + q_Z^* \text{Exc}(\rho) + E_Z.$$

Since $|L_1|$ is base-point free, one can choose an irreducible component G of a general divisor in $|L_1|$ such that G is not $(p \circ p_Z)$ -exceptional. Then $D := p(G)$ is a prime divisor on X . By the same

argument as in the proof of Proposition 3.2, one obtains

$$S_X(B) \geq \frac{n-1}{3(n+1)} \sum_{i=0}^{e-1} (H_Z^{n-i} \cdot L_2^i).$$

Let F be a general fiber of q_Z , and $m := ((2H_Z)|_F)^{n-d}$. Then by the projection formula, one has

$$(H_Z^{n-e+1} \cdot L_2^{e-1}) \geq ((H_Z)^{n-d} \cdot L_Z^{d-e+1} \cdot L_2^{e-1}) = \frac{m(\rho^* L_Y^{d-e+1} \cdot \pi_2^* \mathcal{O}_{\mathbb{P}^{e-1}}(1)^{e-1})}{2^{n-d}} \geq \frac{1}{2^{e-2}}, \quad (3.2)$$

where we use the fact that $n-d = e-2$. Combining Equation (3.2), the Khovanskii–Teissier inequalities and the fact that $(H_Z^n) = 3$, one obtains

$$(H_Z^{n-i} \cdot L_2^i) \geq 3 \left(\frac{1}{3 \cdot 2^{e-2}} \right)^{\frac{i}{e-1}}, \quad 0 \leq i \leq e-1.$$

For $i = 1$, the factor appearing on the right-hand side satisfies

$$\left(\frac{1}{3 \cdot 2^{e-2}} \right)^{\frac{1}{e-1}} = \frac{1}{2} \left(\frac{2}{3} \right)^{\frac{1}{e-1}} \geq \frac{1}{2} \left(\frac{2}{3} \right)^{\frac{1}{2}} = \frac{\sqrt{6}}{6},$$

where the inequality follows from $e \geq 3$. Similarly, for $i = 2$, the factor appearing on the right-hand side satisfies

$$\left(\frac{1}{3 \cdot 2^{e-2}} \right)^{\frac{2}{e-1}} = \frac{1}{4} \left(\frac{2}{3} \right)^{\frac{2}{e-1}} \geq \frac{1}{4} \left(\frac{2}{3} \right) = \frac{1}{6}.$$

Consequently, for every $n \geq 5$ and $e \geq 3$, these estimates yield

$$S_X(B) \geq 3 \cdot \frac{2}{9} \left(1 + \frac{\sqrt{6}}{6} + \frac{1}{6} \right) \geq \frac{7 + \sqrt{6}}{9} > 1 = A_X(B),$$

contradicting the K-semistability of X . □

4. PROOF OF THE MAIN THEOREM

We retain the notation introduced in Section 3. In this section, we prove the following result and then deduce the main theorems.

Theorem 4.1. *Let X be a K-semistable $\mathbb{Q}$ -Gorenstein degeneration of smooth cubic hypersurfaces in $\mathbb{P}^{n+1}$. Then X is isomorphic to a cubic hypersurface in $\mathbb{P}^{n+1}$.*

Proof. If L is a Cartier divisor, then [Fuj90, Corollary 1.7] implies that X is isomorphic to a cubic hypersurface. Thus, we may assume that L is not Cartier. By Proposition 3.1, the image Y of the rational map defined by $|L|$ satisfies $\dim Y = n$. Since Y is integral and nondegenerate, we have

$$2 \leq \deg Y \leq \deg(\nu \circ q) \deg Y = (L_W^n) \leq (p^* L^n) = 3,$$

and hence $\nu \circ q$ is birational and $\deg Y = 2$ or 3 .

First, assume that Y is a quadric hypersurface. Then $\text{rk} Y \geq 3$, Y is normal, and $K_Y \sim -nL_Y$. By Theorem 2.7(2), $D := q_*(2E)$ is an effective Weil divisor. Since $E = p^* L - L_W = p^* L - q^* L_Y$ is q -nef and effective, if D were zero, then E would be q -exceptional. The negativity lemma would give $E = 0$, and hence $p^* L = L_W$. However, $(p^* L)^n = (L^n) = 3$, contradicting the assumption that Y is a quadric. Therefore, D cannot be zero. It follows from Equation (3.1) and the projection formula that

$$3 \geq (p^* L \cdot L_W^{n-1}) = ((L_W + E) \cdot L_W^{n-1}) = (L_W^n) + (E \cdot L_W^{n-1}) = 2 + \frac{1}{2} \deg D, \quad (4.1)$$

where $\deg D$ is the degree of D as a cycle in $\mathbb{P}^{n+1}$. In particular, $\deg D = 1$ or 2 .

We first assume that $\operatorname{rk} Y \geq 5$. Then $\operatorname{Cl}(Y) = \operatorname{Pic}(Y) = \mathbb{Z}[L_Y]$, and we can write $D \sim mL_Y$ for some $m \in \mathbb{Z}_{>0}$. Since $(L_Y^n) = 2$, we must have $m = 1$, $D \sim L_Y$, and $\deg D = 2$. Thus, by Equation (3.1), we obtain $a_0 = a_1 = \cdots = a_{n-1} = 3$ and $a_n = 2$.

Let x be a point at which L is not Cartier, and take the index-one cover with respect to L ,

$$\phi : (\tilde{x} \in \tilde{X}) \rightarrow (x \in X).$$

Then, by Lemma 2.9, Proposition 3.2, and the K-semistability of X , we have

$$\begin{aligned} \operatorname{lct}(\tilde{X}; \mathfrak{m}_{\tilde{x}}) &\geq \operatorname{lct}(\tilde{X}; \mathfrak{J}) = \inf_w \frac{A_{\tilde{X}}(w)}{w(\mathfrak{J})} = \inf_v \frac{A_X(v)}{v(E)} \\ &\geq \inf_v \frac{S_X(v)}{v(E)} \geq \kappa = \frac{(n-1)(3n+2)}{3(n+1)} = n-1 - \frac{n-1}{3(n+1)}. \end{aligned} \quad (4.2)$$

By [Shi17, Proposition 3.13], we have $\operatorname{mult}_{\tilde{x}} \tilde{X} \leq 2$. Abhyankar's multiplicity inequality then implies that $\operatorname{edim}(\tilde{x}, \tilde{X}) \leq n+1$, and hence $\tilde{x} \in \tilde{X}$ is lci. This contradicts Lemma 2.8(2).

If $\operatorname{rk} Y = 4$, then $\operatorname{Cl}(Y) = \mathbb{Z}[L_1] \oplus \mathbb{Z}[L_2]$ and $L_Y \sim L_1 + L_2$, where L_1 and L_2 are the two rulings. Since X is Gorenstein and canonical, $F := q_*(K_W - p^*K_X)$ is an effective Weil divisor whose support is contained in $\operatorname{Supp} D$ and which satisfies

$$F \sim_{\mathbb{Q}} -q_*(nL_W - (n-1)p^*L) \sim_{\mathbb{Q}} \frac{n-1}{2}D - L_Y. \quad (4.3)$$

Write $D \sim aL_1 + bL_2$, where $a, b \geq 0$ and $1 \leq a+b \leq 2$ by (4.1). If $b = 0$, then $[D] \in \mathbb{Z}[L_1]$, and consequently $[F] \in \mathbb{Z}[L_1]$ because $\operatorname{Supp} F \subseteq \operatorname{Supp} D$. However, this contradicts Equation (4.3). The case $a = 0$ is identical, and hence $a = b = 1$ and $D \sim L_Y$. The same argument as in the case $\operatorname{rk} Y \geq 5$ then yields a contradiction.

If $\operatorname{rk} Y = 3$, then Y is an iterated cone over a smooth conic. There exists a family of hyperplane sections of Y covering Y , each of which has a component of multiplicity two. Consequently, for a general member of this family, the corresponding member of $|L|$ has coefficient at least 2 along a prime divisor on X , and hence

$$\alpha(X) \leq \frac{1}{2(n-1)} < \frac{1}{n+1},$$

contradicting the K-semistability of X .

Therefore, Y must be an integral cubic hypersurface in $\mathbb{P}^{n+1}$. In particular, Y is reduced and satisfies S_2 . We claim that $E = 0$ and that $X \dashrightarrow Y$ is an isomorphism. Suppose, to the contrary, that $E \neq 0$. Since p^*L and L_W are both nef, the intersection numbers $(E \cdot (p^*L)^{n-1-i} \cdot L_W^i)$ are nonnegative for every $0 \leq i \leq n-1$. Since $p^*L + L_W$ is ample, we have

$$0 < (E \cdot (p^*L + L_W)^{n-1}) = \sum_{i=0}^{n-1} \binom{n-1}{i} (E \cdot (p^*L)^{n-1-i} \cdot L_W^i).$$

Hence $(E \cdot (p^*L)^{n-1-i} \cdot L_W^i) > 0$ for some i . It follows that

$$0 = (p^*L^n) - (L_W^n) = \sum_{i=0}^{n-1} (E \cdot (p^*L)^{n-1-i} \cdot L_W^i) > 0,$$

a contradiction. Therefore, $E = 0$, $\phi_L : X \rightarrow Y$ is a morphism, and $L = \phi_L^*L_Y$ is Cartier. By [Fuj90, Corollary 1.7], X is isomorphic to a cubic hypersurface in $\mathbb{P}^{n+1}$. $\square$

We now finish the proof of Theorem 1.1 and Corollary 1.2.

Proof of Theorem 1.1. The proofs of (1) and (2) parallel [LX19, Proof of Theorem 1.1] and [Liu22, Proof of Theorem 1.1], so we only sketch the modifications. By Theorem 4.1, every K -semistable Fano variety appearing in the boundary of $\mathcal{M}_{\text{cubic}}^K$ is a cubic n -fold, analogous to [LX19, Lemma 3.17] and [Liu22, Theorem 4.9]. The rest of the argument is the same.

For (3), a direct computation gives $\text{Ext}^2(\Omega_X, \mathcal{O}_X) = 0$ for any cubic n -fold X . Therefore, the $\mathbb{Q}$ -Gorenstein deformations of X are unobstructed, and $\mathcal{M}_{n,V}^K$ is smooth along $\mathcal{M}_{\text{cubic}}^K$. Hence $\mathcal{M}_{\text{cubic}}^K$ is a smooth connected component of $\mathcal{M}_{n,V}^K$. $\square$

Proof of Corollary 1.2. A direct Hilbert–Mumford calculation shows that every cubic fivefold with at worst ordinary double points is GIT stable. Therefore, by Theorem 1.1, such fivefolds are K -stable. The existence of Kähler–Einstein metrics on smooth cubic n -folds follows from [CDS15, Tia15, LX22]. $\square$

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